Tucker's conjecture on asymmetric 2-colorings of locally finite graphs

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The *degree* of a permutation is the number of elements it does not fix. The *minimal degree* of a permutation group G is the minimum of the degrees of the non-identity elements of G. The *motion* of a structure X (such as a graph) is the minimal degree of its automorphism group, $\mathrm{Aut}(X)$. An *asymmetric coloring* (AC) for a permutation group G acting on a set Ω is a coloring of Ω that is not fixed by any non-identity element of G. An asymmetric 2-coloring (A2C) is an AC with 2 colors. An AC (A2C) for a structure X is an AC (A2C, resp.) for $\mathrm{Aut}(X)$.

Tom Tucker proposed the following "Infinite Motion Conjecture" in 2011: *Every connected locally finite graph with infinite motion admits an A2C.*

In 2018, Florian Lehner, Monika Pilśniak, and Marcin Stawiski confirmed the conjecture for graphs of degree $\leqslant 5$. In 2021 we confirmed the conjecture in full generality. We shall sketch some of the ingredients of the proof.

The problem reduces to the analogous question for the *inverse limit of a sequence of finite permutation groups with disjoint domains.* The latter reduces to questions about finite groups, and results by Gluck (1983) and Seress (1997) about A2Cs of primitive permutation groups become relevant. As an illustration, here is the central technical lemma.

Lemma. *Let G be a permutation group acting on a finite set Ω. Let $\varphi : G \twoheadrightarrow T$ be an epimorphism onto a nonabelian simple group T. Then there exists a 2-coloring γ of Ω such that $\varphi(G_\gamma)$ is a proper subgroup of T, where G_γ denotes the subgroup of G that fixes γ.*

The following open question addresses the local cost of an A2C.

Question. Does every connected locally finite *rooted* graph with infinite motion admit a red/blue A2C such that on every sphere about the root, only a bounded number of vertices are colored red?